Local Weyl modules and cyclicity of tensor products for Yangians of G_2

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Abstract

Let $\mathfrak{g}$ be the exceptional complex simple Lie algebra of type G_2. We provide a concrete cyclicity condition for the tensor product of fundamental representations of the Yangian $Y(\mathfrak{g})$. Using this condition, we show that every local Weyl module is isomorphic to an ordered tensor product of fundamental representations of $Y(\mathfrak{g})$.

Key words: Yangian; G_2; Local Weyl modules; Cyclicity condition

1 Introduction

There is a rich structure theory behind the finite-dimensional representations of Yangians $Y(\mathfrak{g})$ and quantum affine algebras $U_q(\hat{\mathfrak{g}})$ as the category of their finite-dimensional representations is not semi-simple, where $\mathfrak{g}$ is a complex simple Lie algebra of rank l. Finite-dimensional irreducible representations of $U_q(\hat{\mathfrak{g}})$ are parameterized by l-tuples of polynomials $\mathrm{P} = \big(P_1(u), \dots, P_l(u)\big)$, where $P_i(0) = 1$, see [6]. In [9], V. Chari and A. Pressley showed that in the class of all highest weight representations associated to P, there is a unique (up to isomorphism) finite-dimensional highest weight representation $W(\mathrm{P})$ such that any other representation in this class is a quotient of $W(\mathrm{P})$. It has been established that $W(\mathrm{P})$ is isomorphic to an ordered tensor product of fundamental representations of $U_q(\hat{\mathfrak{g}})$, and a proof of this fact can be found in [3]. In [2], V. Chari provided a method to find a concrete cyclicity condition for an ordered tensor product of Kirillov-Reshetikhin modules using braided group action on the imaginary root vectors.

The finite-dimensional representation theory of $Y(\mathfrak{g})$ is an analogue of the one of $U_q(\hat{\mathfrak{g}})$. Let π be a l-tuple of polynomials. One can define the local Weyl module of the Yangian $Y(\mathfrak{g})$ similarly. We gave the definition of local Weyl module $W(\pi)$ by generators and defining relations in [12], and proved that $W(\pi)$ is isomorphic to an ordered tensor product of fundamental representations when $\mathfrak{g}$ is classical. The main challenge in [12] was to find an explicit cyclicity condition for an ordered tensor product, and the methodology will be introduced in Section 2 for more detail.

This work is a continuation of the paper [12]. We introduce a new algorithm to compute certain associated polynomials when $\mathfrak{g}$ is of type G_2, which enables us to provide an explicit cyclicity condition for an ordered tensor product of fundamental representations, see Theorem 1. Using this condition, we show that the local Weyl module $W(\pi)$ is isomorphic to an ordered tensor product of fundamental representations in Theorem 2.

2 Preliminary

In this section, we give a brief review of the previous paper [12] and one question left open in *loc. cit.*

Definition 2.1. *Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with rank l and let $A = (a_{ij})_{i,j\in I}$, where $I = \{1, 2, \dots, l\}$, be its Cartan matrix. Let $D = \mathrm{diag}\,(d_1, \dots, d_l)$, $d_i \in \mathbb{N}$, such that $d_1, d_2, \dots, d_l$ are co-prime and DA is symmetric. The Yangian $Y(\mathfrak{g})$ is defined to be the associative algebra with generators $x^{\pm}_{i,r}$, $h_{i,r}$, $i \in I$, $r \in \mathbb{Z}_{\geq 0}$, and the following*

defining relations:

$$[h_{i,r}, h_{j,s}] = 0, \qquad [h_{i,0}, x^{\pm}_{j,s}] = \pm d_i a_{ij} x^{\pm}_{j,s}, \qquad [x^{+}_{i,r}, x^{-}_{j,s}] = \delta_{i,j} h_{i,r+s},$$

$$[h_{i,r+1}, x^{\pm}_{j,s}] - [h_{i,r}, x^{\pm}_{j,s+1}] = \pm\frac{1}{2} d_i a_{ij} (h_{i,r} x^{\pm}_{j,s} + x^{\pm}_{j,s} h_{i,r}),$$

$$[x^{\pm}_{i,r+1}, x^{\pm}_{j,s}] - [x^{\pm}_{i,r}, x^{\pm}_{j,s+1}] = \pm\frac{1}{2} d_i a_{ij} (x^{\pm}_{i,r} x^{\pm}_{j,s} + x^{\pm}_{j,s} x^{\pm}_{i,r}),$$

$$\sum_{\pi} [x^{\pm}_{i,r_{\pi(1)}}, [x^{\pm}_{i,r_{\pi(2)}}, \ldots, [x^{\pm}_{i,r_{\pi(m)}}, x^{\pm}_{j,s}] \cdots]] = 0, i \neq j,$$

for all sequences of non-negative integers $r_1, \ldots, r_m$, where $m = 1 - a_{ij}$ and the sum is over all permutations π of $\{1, \ldots, m\}$.

Denote by $V_m(a)$, $m \geq 1$ and $a \in \mathbb{C}$, the finite-dimensional irreducible representation of $Y(\mathfrak{sl}_2)$ associated to the Drinfeld polynomial $\big(u-a\big)\big(u-(a+1)\big)\ldots\big(u-(a+m-1)\big)$.

Proposition 2.2 (Proposition 3.5,[8])**.** *The module $V_m(a)$ has a basis $\{w_0, w_1, \ldots, w_m\}$ on which the action of $Y(\mathfrak{sl}_2)$ is given by*

$$\begin{gathered} x^{+}_k w_s = (s+a)^k (s+1) w_{s+1}, \quad x^{-}_k w_s = (s+a-1)^k (m-s+1) w_{s-1}, \\ h_k w_s = \big((s+a-1)^k s(m-s+1) - (s+a)^k (s+1)(m-s)\big) w_s. \end{gathered}$$

Suppose that a fixed reduced expression of the longest element of the Weyl group of $\mathfrak{g}$ is $w_0 = s_{r_1} s_{r_2} \ldots s_{r_p}$, where s_{r_j}, for $1 \leq j \leq p$, are simple reflections. Denote the i-th fundamental weight by ω_i and the i-th fundamental representation of $Y(\mathfrak{g})$ by $V_a(\omega_i)$. Let v^+ and v^- be the highest and lowest weight vectors in the fundamental representation $V_a(\omega_i)$, respectively. Suppose $s_{r_{j+1}} s_{r_{j+2}} \ldots s_{r_p}(\omega_i) = m_j \omega_{r_j} + \sum\limits_{n \neq r_j} c_n \omega_n$. Then

$$v^- = \big(x^{-}_{r_1,0}\big)^{m_1} \big(x^{-}_{r_2,0}\big)^{m_2} \ldots \big(x^{-}_{r_p,0}\big)^{m_p} v^+.$$

Define $s_{r_{j+1}} s_{r_{j+2}} \ldots s_{r_p}$ by σ_j and $(x^{-}_{r_{j+1},0})^{m_{j+1}} (x^{-}_{r_{j+2},0})^{m_{j+2}} \ldots (x^{-}_{r_p,0})^{m_p} v^+$ by $v_{\sigma_j(\omega_i)}$. Let Y_i be the subalgebra generated by $\{x^{\pm}_{i,r}, h_{i,r} | r \in \mathbb{Z}_{\geq 0}\}$ for $i \in I$, which is isomorphic to $Y(\mathfrak{sl}_2)$. Denote by $Y_{r_j}(v_{\sigma_j(\omega_i)})$ the Y_{r_j}-module generated by the extremal vector $v_{\sigma_j(\omega_i)}$. We remark that it has been established in Section 5 in [12] that $Y_{r_j}(v_{\sigma_j(\omega_{b_m})})$ is a highest weight representation and the degree of its associated polynomial is m_j. In [12], we showed:

Theorem 5.2 [12] An ordered tensor product $L = V_{a_1}(\omega_{b_1}) \otimes V_{a_2}(\omega_{b_2}) \otimes \ldots \otimes V_{a_k}(\omega_{b_k})$ is a highest weight representation if for all $1 \leq j \leq p$ and $1 \leq m < n \leq k$, when $b_n = r_j$, the difference of the number $\frac{a_n}{d_{r_j}}$ and any root of the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{b_m})})$ does not equal 1.

We proved the above theorem by adopting the ideas in [2]. To find an explicit cyclicity condition, it is enough to compute the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{b_m})})$ for $1 \leq j \leq p$ and $1 \leq m \leq l$. When $\mathfrak{g}$ is a classical simple Lie algebra, by computing the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{b_1})})$ using some of the defining relations of $Y(\mathfrak{g})$, a concrete cyclicity condition of L was obtained in Theorem 5.18 in [12].

For the following reasons, it becomes complicated if one tries to describe a concrete cyclicity condition for the tensor product when $\mathfrak{g}$ is an exceptional simple Lie algebra . In this case, m_j may be greater than or equal to 3. The computations of the eigenvalue of $h_{r_j,k}$ on the weight vector $v_{\sigma_j(\omega_i)}$, $3 \leq k \leq m_j$, are tedious if one uses defining relations of $Y(\mathfrak{g})$. In addition, the path from the highest weight vector v^+ to the lowest one v^- is more subtle than in the case when $\mathfrak{g}$ is a classical simple Lie algebra. For instance, when $\mathfrak{g}$ is of type F_4, in $V_a(\omega_2)$, a path is:

$$\begin{aligned} v^- &= x^-_{2,0}x^-_{1,0}(x^-_{3,0})^2(x^-_{2,0})^2x^-_{1,0}(x^-_{3,0})^2x^-_{2,0}(x^-_{4,0})^4(x^-_{3,0})^4(x^-_{2,0})^3(x^-_{1,0})^3(x^-_{3,0})^2 \\ &\quad (x^-_{2,0})^2(x^-_{3,0})^2(x^-_{4,0})^4(x^-_{3,0})^2(x^-_{2,0})^2x^-_{1,0}(x^-_{3,0})^2x^-_{2,0}v^+. \end{aligned}$$

It seems to us that step-by-step computations are needed to compute the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{b_1})})$. The amount of time required to compute these associated polynomials would be tremendous using exactly the same approach as in [12].

3 A concrete cyclicity condition and local Weyl modules

From now on, let $\mathfrak{g}$ denote the simple Lie algebra of type G_2, unless the contrary is stated. Let α_1 and α_2 be the simple long and short roots, respectively (as labelled in [1]) and let ω_1 and ω_2 be the fundamental weights. The Cartan matrix of $\mathfrak{g}$ is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$. Let $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$, so DA is symmetric. The Weyl group $\mathcal{W}$ of $\mathfrak{g}$ is generated by s_1 and s_2 such that $s_1(\alpha_1) = -\alpha_1$, $s_2(\alpha_1) = \alpha_1 + 3\alpha_2$, $s_1(\alpha_2) = \alpha_1 + \alpha_2$, and $s_2(\alpha_2) = -\alpha_2$. A reduced expression of the longest element in Weyl group is $w_0 = s_1s_2s_1s_2s_1s_2$. Using the method indicated in Section 2 of this paper, we found a path form v^+ to v^-.

Lemma 3.1. *Let $a \in \mathbb{C}$.*

1. *In $V_a(\omega_1)$, $v^- = x^-_{1,0}(x^-_{2,0})^3(x^-_{1,0})^2(x^-_{2,0})^3x^-_{1,0}v^+$.*

2. *In $V_a(\omega_2)$, $v^- = x^-_{2,0}x^-_{1,0}(x^-_{2,0})^2x^-_{1,0}x^-_{2,0}v^+$.*

Note that $x^{\pm}_{1,r}, h_{1,r}$ do not satisfy the defining relations of $Y(\mathfrak{sl}_2)$. Therefore, we need to re-scale the generators. Let $\tilde{x}^{\pm}_{1,r} = \frac{\sqrt{3}}{3^{r+1}} x^{\pm}_{1,r}$ and $\tilde{h}_{1,r} = \frac{1}{3^{r+1}} h_{1,r}$, then $\tilde{x}^{\pm}_{1,r}, \tilde{h}_{1,s}$ satisfy the defining relations of $Y(\mathfrak{sl}_2)$. Now we are in the position to compute the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{b_m})})$, where $b_m \in I = \{1, 2\}$.

Proposition 3.2. *Let $v^+ \in V_a(\omega_1)$.*

Item	$Y(\mathfrak{sl}_2)$*-module*	*Associated polynomial*
1	$Y_1(v^+)$	$u - \frac{a}{3}$
2	$Y_2(x^-_{1,0}v^+)$	$\big(u - (a + \frac{3}{2})\big)\big(u - (a + \frac{1}{2})\big)\big(u - (a - \frac{1}{2})\big)$
3	$Y_1\big((x^-_{2,0})^3 x^-_{1,0} v^+\big)$	$\big(u - \frac{a+2}{3}\big)\big(u - \frac{a+1}{3}\big)$
4	$Y_2\big((x^-_{1,0})^2 (x^-_{2,0})^3 x^-_{1,0} v^+\big)$	$\big(u - (a + \frac{7}{2})\big)\big(u - (a + \frac{5}{2})\big)\big(u - (a + \frac{3}{2})\big)$
5	$Y_1\big((x^-_{2,0})^3 (x^-_{1,0})^2 (x^-_{2,0})^3 x^-_{1,0} v^+\big)$	$u - \frac{a+3}{3}$

Proof. We omit the proof of items 1, 2 and 5 since these proofs can be checked using the same approach as in paper [12]. Lemma 3.6 is devoted to proving the third item. The fourth item is proved in Lemma 3.8. $\square$

Let H be the subalgebra of $Y(\mathfrak{g})$ generated by all $h_{i,k}$ and $h_i(u) = 1 + h_{i,0}u^{-1} + h_{i,1}u^{-2} + \ldots$, where $i \in I = \{1, 2\}$ and $k \in \mathbb{Z}_{\geq 0}$. We use alternate generators for H as given in [10]. Let

$$H_i(u) = \sum_{k=0}^{\infty} H_{i,k} u^{-k-1} := \ln\big(h_i(u)\big).$$

An explicit computation shows that

$$H_i(u) = h_{i,0}u^{-1} + \big(h_{i,1} - \frac{1}{2}(h_{i,0})^2\big)u^{-2} + \big(h_{i,2} - h_{i,0}h_{i,1} + \frac{1}{3}(h_{i,0})^3\big)u^{-3} + \ldots. \tag{3.1}$$

Lemma 3.3 (Corollary 1.5, [10])**.** *Let $\mathfrak{g}$ be a complex simple Lie algebra.*

$$[H_{i,k}, x^{\pm}_{j,l}] = \pm\, d_i a_{ij} x^{\pm}_{j,l+k} \pm \sum_{\substack{0 \leq s \leq k-2 \\ k+s \ \ even}} 2^{s-k} (d_i a_{ij})^{k+1-s} \frac{\binom{k+1}{s}}{k+1} x^{\pm}_{j,l+s}.$$

Let V be a finite-dimensional highest weight representation of $Y(\mathfrak{sl}_2)$ whose associated polynomial is π. Let v^+ and v^- be highest and lowest weight vectors of V, respectively.

Lemma 3.4. $h(u)v^- = \frac{\pi(u-1)}{\pi(u)} v^-$.

Proof. It is enough to consider the case when V is irreducible. It was established in Proposition 3.1 [5] that $h(u)v^- = \frac{p(u)}{p(u+1)} v^-$, where $p(u)$ is the associated polynomial of the right dual V^t. It was showed in Proposition 2.4 in [8] that $p(u) = \pi(u-1)$. Therefore this lemma is proved. $\square$

To show Lemma 3.6, we need the following corollary. We remark that in the proof of the corollary, we use some algorithm which did not show up in [12].

Corollary 3.5. *In the representation* $Y_2(x_{1,0}^- v^+)$,

1. $$\begin{aligned}&\big(x_{2,1}^-(x_{2,0}^-)^2 + x_{2,0}^- x_{2,1}^- x_{2,0}^- + (x_{2,0}^-)^2 x_{2,1}^-\big)x_{1,0}^- v^+\\ &\qquad = \Big((a+\frac{3}{2}) + (a+\frac{1}{2}) + (a-\frac{1}{2})\Big)(x_{2,0}^-)^3 x_{1,0}^- v^+.\end{aligned}$$
2. $$\begin{aligned}&\big(x_{2,2}^-(x_{2,0}^-)^2 + x_{2,0}^- x_{2,2}^- x_{2,0}^- + (x_{2,0}^-)^2 x_{2,2}^-\big)x_{1,0}^- v^+\\ &\qquad = \Big((a+\frac{3}{2})^2 + (a+\frac{1}{2})^2 + (a-\frac{1}{2})^2\Big)(x_{2,0}^-)^3 x_{1,0}^- v^+.\end{aligned}$$

Proof. It follows from Proposition 3.2 that the associated polynomial of the highest weight representation $Y_2(x_{1,0}^- v^+)$ is $\big(u-(a+\frac{3}{2})\big)\big(u-(a+\frac{1}{2})\big)\big(u-(a-\frac{1}{2})\big)$. Therefore, $h_2(u)(x_{1,0}^- v^+) = \frac{u-(a-\frac{3}{2})}{u-(a+\frac{3}{2})} x_{1,0}^- v^+$ and $h_2(u)\big((x_{2,0}^-)^3 x_{1,0}^- v^+\big) = \frac{u-(a+\frac{5}{2})}{u-(a-\frac{1}{2})}(x_{2,0}^-)^3 x_{1,0}^- v^+$ by Lemma 3.4. Thus

$$H_2(u)(x_{1,0}^- v^+) = \Big(\ln(1-(a-\frac{3}{2})u^{-1}) - \ln(1-(a+\frac{3}{2})u^{-1})\Big)x_{1,0}^- v^+$$

and

$$H_2(u)\big((x_{2,0}^-)^3 x_{1,0}^- v^+\big) = \Big(\ln(1-(a+\frac{5}{2})u^{-1}) - \ln(1-(a-\frac{1}{2})u^{-1})\Big)(x_{2,0}^-)^3 x_{1,0}^- v^+.$$

In particular, we have both $H_{2,1}(x_{2,0}^-)^3 x_{1,0}^- v^+ = \frac{1}{2}\big((a-\frac{1}{2})^2-(a+\frac{5}{2})^2\big)(x_{2,0}^-)^3 x_{1,0}^- v^+$ and $H_{2,2}(x_{2,0}^-)^3 x_{1,0}^- v^+ = \frac{1}{3}\big((a-\frac{1}{2})^3-(a+\frac{5}{2})^3\big)(x_{2,0}^-)^3 x_{1,0}^- v^+$. By Lemma 3.3, $[H_{2,1}, x_{2,0}^-] = -2x_{2,1}^-$ and $[H_{2,2}, x_{2,0}^-] = -2x_{2,2}^- - \frac{2}{3}x_{2,0}^-$. We are going to show the first item in this corollary.

$$\begin{aligned}&H_{2,1}(x_{2,0}^-)^3 x_{1,0}^- v^+\\ &\quad = [H_{2,1}, (x_{2,0}^-)^3]x_{1,0}^- v^+ + (x_{2,0}^-)^3 H_{2,1} x_{1,0}^- v^+\\ &\quad = -2\big(x_{2,1}^-(x_{2,0}^-)^2 + x_{2,0}^- x_{2,1}^- x_{2,0}^- + (x_{2,0}^-)^2 x_{2,1}^-\big)x_{1,0}^- v^+\\ &\quad + \frac{1}{2}\big((a+\frac{3}{2})^2 - (a-\frac{3}{2})^2\big)(x_{2,0}^-)^3 x_{1,0}^- v^+.\end{aligned}$$

Therefore,

$$\begin{aligned}&\big(x_{2,1}^-(x_{2,0}^-)^2 + x_{2,0}^- x_{2,1}^- x_{2,0}^- + (x_{2,0}^-)^2 x_{2,1}^-\big)x_{1,0}^- v^+\\ &\qquad = \frac{1}{4}\big((a+\frac{5}{2})^2 - (a-\frac{1}{2})^2 + (a+\frac{3}{2})^2 - (a-\frac{3}{2})^2\big)(x_{2,0}^-)^3 x_{1,0}^- v^+\\ &\qquad = (3a+\frac{3}{2})(x_{2,0}^-)^3 x_{1,0}^- v^+.\end{aligned}$$

The second item in this corollary can be obtained similarly, so we omit the proof. $\square$

Lemma 3.6. *The associated polynomial of the representation* $Y_1\Big((x^-_{2,0})^3x^-_{1,0}v^+\Big)$ *is given by* $\big(u-\frac{a+2}{3}\big)\big(u-\frac{a+1}{3}\big)$.

Proof. The associated polynomial of the representation $Y_1\Big((x^-_{2,0})^3x^-_{1,0}v^+\Big)$ is of degree 2, say $(u-a_1)(u-a_2)$. The eigenvalues of $(x^-_{2,0})^3x^-_{1,0}v^+$ under $\tilde{h}_{1,1}$ and $\tilde{h}_{1,2}$ will tell us the values of a_1 and a_2. We first compute the eigenvalues of $(x^-_{2,0})^3x^-_{1,0}v^+$ under $H_{1,1}$ and $H_{1,2}$.

$$\begin{aligned}
&H_{1,1}(x^-_{2,0})^3x^-_{1,0}v^+\\
&\quad=[H_{1,1},x^-_{2,0}](x^-_{2,0})^2x^-_{1,0}v^+ + x^-_{2,0}[H_{1,1},x^-_{2,0}]x^-_{2,0}x^-_{1,0}v^+\\
&\qquad+(x^-_{2,0})^2[H_{1,1},x^-_{2,0}]x^-_{1,0}v^+ + (x^-_{2,0})^3H_{1,1}x^-_{1,0}v^+\\
&\quad=3x^-_{2,1}(x^-_{2,0})^2x^-_{1,0}v^+ + 3x^-_{2,0}x^-_{2,1}x^-_{2,0}x^-_{1,0}v^+ + 3(x^-_{2,0})^2x^-_{2,1}x^-_{1,0}v^+\\
&\qquad+\frac{1}{2}\big((a-3)^2-a^2\big)(x^-_{2,0})^3x^-_{1,0}v^+\\
&\quad=3\big((a-\frac{1}{2})+(a+\frac{1}{2})+(a+\frac{3}{2})-(a+\frac{3}{2})\big)(x^-_{2,0})^3x^-_{1,0}v^+\\
&\quad=6a(x^-_{2,0})^3x^-_{1,0}v^+,
\end{aligned}$$

where the third equality follows from the first item of Corollary 3.5.

$$\begin{aligned}
&H_{1,2}(x^-_{2,0})^3x^-_{1,0}v^+\\
&=[H_{1,2},x^-_{2,0}](x^-_{2,0})^2x^-_{1,0}v^+ + x^-_{2,0}[H_{1,2},x^-_{2,0}]x^-_{2,0}x^-_{1,0}v^+\\
&\quad+(x^-_{2,0})^2[H_{1,2},x^-_{2,0}]x^-_{1,0}v^+ + (x^-_{2,0})^3H_{1,2}x^-_{1,0}v^+\\
&=\big(3x^-_{2,2}+(\frac{3}{2})^2x^-_{2,0}\big)(x^-_{2,0})^2x^-_{1,0}v^+ + x^-_{2,0}\big(3x^-_{2,2}+(\frac{3}{2})^2x^-_{2,0}\big)x^-_{2,0}x^-_{1,0}v^+\\
&+(x^-_{2,0})^2\big(3x^-_{2,2}+(\frac{3}{2})^2x^-_{2,0}\big)x^-_{1,0}v^+ - \frac{1}{3}\big((a-3)^3-a^3\big)(x^-_{2,0})^3x^-_{1,0}v^+\\
&=3\big((a-\frac{1}{2})^2+(a+\frac{1}{2})^2+(a+\frac{3}{2})^2+\frac{9}{4}-(a^2+3a+3)\big)(x^-_{2,0})^3x^-_{1,0}v^+\\
&=(6a^2+6)(x^-_{2,0})^3x^-_{1,0}v^+.
\end{aligned}$$

It follows from equation (3.1) and the above computations that

$$\begin{aligned}
\tilde{h}_{1,1}(x^-_{2,0})^3x^-_{1,0}v^+ &= (\tfrac{2a}{3}+2)(x^-_{2,0})^3x^-_{1,0}v^+;\\
\tilde{h}_{1,2}(x^-_{2,0})^3x^-_{1,0}v^+ &= (2(\tfrac{a}{3})^2+\tfrac{4a}{3}+\tfrac{14}{9})(x^-_{2,0})^3x^-_{1,0}v^+.
\end{aligned}$$

It follows from relation (5.1) in [12] that $a_1+a_2=\frac{2a}{3}+1$ and $a_1^2+a_2^2+a_1+a_2=2(\frac{a}{3})^2+\frac{4a}{3}+\frac{14}{9}$. Then $a_1=\frac{a+1}{3}$ and $a_2=\frac{a+2}{3}$, or vice-versa with a_1 and a_2 switched. Thus the associated polynomial of the representation $Y_1\Big((x^-_{2,0})^3x^-_{1,0}v^+\Big)$ is $\big(u-\frac{a+2}{3}\big)\big(u-\frac{a+1}{3}\big)$. □

Similar to the proof of Corollary 3.5, we can prove the following corollary. We remark that: $h_1(u)\big((x_{2,0}^-)^3x_{1,0}^-v^+\big) = \frac{u-(a+1)}{u-(a+2)}\frac{u-a}{u-(a+1)}(x_{2,0}^-)^3x_{1,0}^-v^+$.

Corollary 3.7. *In the representation* $Y_1\Big((x_{2,0}^-)^3x_{1,0}^-v^+\Big)$,

1. $(x_{1,1}^-x_{1,0}^- + x_{1,0}^-x_{1,1}^-)(x_{2,0}^-)^3x_{1,0}^-v^+ = \big((a+1)+(a+2)\big)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+$;
2. $(x_{1,2}^-x_{1,0}^- + x_{1,0}^-x_{1,2}^-)(x_{2,0}^-)^3x_{1,0}^-v^+ = \big((a+1)^2+(a+2)^2\big)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+$;
3. $(x_{1,3}^-x_{1,0}^- + x_{1,0}^-x_{1,3}^-)(x_{2,0}^-)^3x_{1,0}^-v^+ = \big((a+1)^3+(a+2)^3\big)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+$.

Lemma 3.8. *The associated polynomial of the module* $Y_2\big((x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\big)$ *is given by* $\big(u-(a+\frac{3}{2})\big)\big(u-(a+\frac{5}{2})\big)\big(u-(a+\frac{7}{2})\big)$.

Proof. The associated polynomial of the representation $Y_2\big((x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\big)$ is of degree 3, say $(u-a_1)(u-a_2)(u-a_3)$. The eigenvalues of $(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+$ under $h_{2,1}$, $h_{2,2}$ and $h_{2,3}$ will tell us the values of a_1, a_2 and a_3.

$$
\begin{aligned}
&H_{2,1}(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= [H_{2,1},x_{1,0}^-]x_{1,0}^-(x_{2,0}^-)^3x_{1,0}^-v^+ + x_{1,0}^-[H_{2,1},x_{1,0}^-](x_{2,0}^-)^3x_{1,0}^-v^+ + (x_{1,0}^-)^2H_{2,1}(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= 3\big(x_{1,1}^-x_{1,0}^- + x_{1,0}^-x_{1,1}^-\big)(x_{2,0}^-)^3x_{1,0}^-v^+ - \frac{1}{2}\Big((a+\frac{5}{2})^2-(a-\frac{1}{2})^2\Big)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= \big((6a+9)-(3a+3)\big)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= (3a+6)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+,
\end{aligned}
$$

where the third equality follows from the first item of Corollary 3.7.

$$
\begin{aligned}
&H_{2,2}(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= [H_{2,2},x_{1,0}^-]x_{1,0}^-(x_{2,0}^-)^3x_{1,0}^-v^+ + x_{1,0}^-[H_{2,2},x_{1,0}^-](x_{2,0}^-)^3x_{1,0}^-v^+\\
&\qquad + (x_{1,0}^-)^2H_{2,2}(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= (3x_{1,2}^- + \frac{9}{4}x_{1,0}^-)x_{1,0}^-(x_{2,0}^-)^3x_{1,0}^-v^+ + x_{1,0}^-(3x_{1,2}^- + \frac{9}{4}x_{1,0}^-)(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\qquad - \frac{1}{3}\Big((a+\frac{5}{2})^3-(a-\frac{1}{2})^3\Big)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= (3a^2+12a+\frac{57}{4})(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+.
\end{aligned}
$$

$$
\begin{aligned}
&H_{2,3}(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+\\
&\quad= [H_{2,3},x_{1,0}^-]x_{1,0}^-(x_{2,0}^-)^3x_{1,0}^-v^+ + x_{1,0}^-[H_{2,3},x_{1,0}^-](x_{2,0}^-)^3x_{1,0}^-v^+\\
&\qquad + (x_{1,0}^-)^2H_{2,3}(x_{2,0}^-)^3x_{1,0}^-v^+
\end{aligned}
$$

$$
\begin{aligned}
&= (3x_{1,3}^- + \frac{27}{4}x_{1,1}^-)x_{1,0}^-(x_{2,0}^-)^3x_{1,0}^-v^+ + x_{1,0}^-(3x_{1,3}^- + \frac{27}{4}x_{1,1}^-)(x_{2,0}^-)^3x_{1,0}^-v^+ \\
&\quad - \frac{1}{4}\big((a+\frac{5}{2})^4 - (a-\frac{1}{2})^4\big)(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+ \\
&= (3a^3 + 18a^2 + \frac{171}{4}a + \frac{75}{2})(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+.
\end{aligned}
$$

It follows from equation (3.1) and the above computations that

$$
\begin{aligned}
h_{2,1}(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+ &= 3(a+\frac{7}{2})(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+; \\
h_{2,2}(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+ &= 3(a+\frac{7}{2})^2(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+; \\
h_{2,3}(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+ &= 3(a+\frac{7}{2})^3(x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\frac{u-(a_1-1)}{u-a_1}\frac{u-(a_2-1)}{u-a_2}\frac{u-(a_3-1)}{u-a_3} \\
&\quad = 1 + 3u^{-1} + (a_1+a_2+a_3+3)u^{-2} + (a_1^2+a_2^2+a_3^2+2a_1+2a_2+2a_3+1)u^{-3} \\
&\qquad + (a_1^3+a_2^3+a_3^3+2a_1^2+2a_2^2+2a_3^2+a_1a_2+a_1a_3+a_2a_3+a_1+a_2+a_3)u^{-4} \\
&\qquad + \dots.
\end{aligned}
$$

Computations show that $a_1 = a+\frac{3}{2}$, $a_2 = a+\frac{5}{2}$ and $a_3 = a+\frac{7}{2}$, or vice-versa with a_1, a_2 and a_3 switched. Thus the associated polynomial of the representation $Y_2((x_{1,0}^-)^2(x_{2,0}^-)^3x_{1,0}^-v^+)$ is given by $\big(u-(a+\frac{3}{2})\big)\big(u-(a+\frac{5}{2})\big)\big(u-(a+\frac{7}{2})\big)$. □

We will omit the proof of the following proposition since it can be proved using some of the defining relations of $Y(\mathfrak{g})$.

Proposition 3.9. *Let $v^+ \in V_a(\omega_2)$.*

Item	$Y(\mathfrak{sl}_2)$*-module*	*Associated polynomial*
1	$Y_2(v^+)$	$u-a$
2	$Y_1(x_{2,0}^-v^+)$	$u-(\frac{a}{3}+\frac{1}{2})$
3	$Y_2\left(x_{1,0}^-x_{2,0}^-v^+\right)$	$\big(u-(a+3)\big)\big(u-(a+2)\big)$
4	$Y_1((x_{2,0}^-)^2x_{1,0}^-x_{2,0}^-v^+)$	$u-(\frac{a}{3}+\frac{7}{6})$
5	$Y_2(x_{1,0}^-(x_{2,0}^-)^2x_{1,0}^-x_{2,0}^-v^+)$	$u-(a+5)$

We summarize all results in Propositions 1 and 2 into the coming corollary. Denote by $T(b_m, r_j)$ the set of all possible roots of the associated polynomial of $Y_{r_j}(v_{\sigma_j(\omega_{b_m})})$.

Corollary 3.10. *$T(1,1) = \{\frac{a}{3}, \frac{a+1}{3}, \frac{a+2}{3}, \frac{a+3}{3}\}$, $T(1,2) = \{a-\frac{1}{2}, a+\frac{1}{2}, a+\frac{3}{2}, a+\frac{5}{2}, a+\frac{7}{2}\}$, $T(2,1) = \{\frac{a}{3}+\frac{1}{2}, \frac{a}{3}+\frac{7}{6}\}$ and $T(2,2) = \{a, a+2, a+3, a+5\}$.*

By Theorem 5.2 in [12] and Proposition 3.8 in [7], we have the following theorem.

Theorem 3.11. *Let $L = V_{a_1}(\omega_{b_1}) \otimes V_{a_2}(\omega_{b_2}) \otimes \ldots \otimes V_{a_k}(\omega_{b_k})$ be an ordered tensor product of fundamental representations of $Y(\mathfrak{g})$, and define $S(1,1) = \{3,4,5,6\}$, $S(1,2) = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}\}$, $S(2,1) = \{\frac{9}{2}, \frac{13}{2}\}$ and $S(2,2) = \{1,3,4,6\}$.*

1. *If $a_j - a_i \notin S(b_i, b_j)$ for $1 \le i < j \le k$, then L is a highest weight representation of $Y(\mathfrak{g})$.*
2. *If $a_j - a_i \notin S(b_i, b_j)$ for $1 \le i \ne j \le k$, then L is an irreducible representation of $Y(\mathfrak{g})$.*

Remark 3.12.

(i). In Section 6.2 [2], V. Chari gave the set $\mathrm{S}(i_1, i_2)$, $i_1 \le i_2$ of values of $a_1^{-1}a_2$ for which the tensor product $V(i_1, a_1) \otimes V(i_2, a_2)$ may fail to be irreducible as a module over the quantum loop algebra. Note that, in the following discussion, we interchange the node labels on the Dynkin diagram of G_2 as used in that paper. V. Chari found that

$$\mathrm{S}(1,1) = \{q^6, q^8, q^{10}, q^{12}\}. \qquad \mathrm{S}(2,1) = \{q^7, q^{11}\}.$$
$$\mathrm{S}(1,2) = \{q^3, q^7\}. \qquad \mathrm{S}(2,2) = \{q^2, q^6, q^8, q^{12}\}.$$

(ii). In this paragraph, let $\mathfrak{g}$ be any finite-dimensional complex simple Lie algebra of rank l. The fundamental representations of $Y(\mathfrak{g})$ can be treated as special cases of the Kirillov-Reshetikhin modules, which are the finite-dimensional irreducible representations associated to an l-tuple of polynomials $\pi = \big(\pi_1(u), \pi_2(u), \ldots, \pi_l(u)\big)$ such that $\pi_i(u) = \big(1 - au^{-1}\big)\big(1 - (a+1)u^{-1}\big)\ldots\big(1 - (a+m-1)u^{-1}\big)$ and $\pi_j(u) = 1$, for all $j \ne i$. The methods used in this paper could shed some light on obtaining a concrete cyclicity condition for the tensor product of Kirillov-Reshetikhin modules of $Y(\mathfrak{g})$.

We close this section by providing the structure of $W(\pi)$. To obtain an upper bound on the dimension of $W(\pi)$, we use the dimension of the local Weyl module $W(\lambda)$ of the current algebra $\mathfrak{g}[t]$, which is given in [11].

Proposition 3.13 (Corollary 9.5, [11])**.** *Let $\lambda = m_1\omega_1 + m_2\omega_2$. Then*

$$\mathrm{Dim}\,\big(W(\lambda)\big) = \Big(\mathrm{Dim}\,\big(W(\omega_1)\big)\Big)^{m_1}\Big(\mathrm{Dim}\,\big(W(\omega_2)\big)\Big)^{m_2}.$$

It follows from Theorem 3.11 and Proposition 2.15 in [5] that

Proposition 3.14. *Let* $\pi = \big(\pi_1(u), \pi_2(u)\big)$ *be a pair of monic polynomials in* u, *and let* $\pi_i(u) = \prod_{j=1}^{m_i}(u - a_{i,j})$. *Let* $S = \{a_{1,1}, \ldots, a_{1,m_1}, a_{2,1}, \ldots, a_{2,m_2}\}$ *be a multi-set of roots. Let* $a_1 = a_{i,j}$ *be one of the numbers in* S *with maximal real part and let* $b_1 = i$. *Similarly, let* $a_r = a_{s,t}(r \geq 2)$ *be one of the numbers in* $S \setminus \{a_1, \ldots, a_{r-1}\}$ *with maximal real part and let* $b_r = s$. *Let* $L = V_{a_1}(\omega_{b_1}) \otimes V_{a_2}(\omega_{b_2}) \otimes \ldots \otimes V_{a_k}(\omega_{b_k})$, *where* $k = m_1 + m_2$. *Then* L *is a highest weight representation and its associated polynomials are* $\pi_1(u)$ *and* $\pi_2(u)$.

Theorem 3.15. *The local Weyl module* $W(\pi)$ *of* $Y(\mathfrak{g})$ *associated to* π *is isomorphic to* L *as in Proposition 3.14.*

Proof. Let $\lambda = m_1\omega_1 + m_2\omega_2$. On the one hand, $\mathrm{Dim}\big(W(\pi)\big) \leq \mathrm{Dim}\big(W(\lambda)\big)$ by Theorem 3.8 [12]; on the other hand, $\mathrm{Dim}\big(W(\pi)\big) \geq \mathrm{Dim}(L)$ by the maximality of the local Weyl modules of Yangians. Note that as G_2-modules, $W(\omega_i) \cong KR(\omega_i) \cong V_a(\omega_i)$ and the latter isomorphism follows easily from the main theorem of Section 2.3 in [4] and Theorem 6.3 in [5]. In particular, $\mathrm{Dim}\big(W(\omega_i)\big) = \mathrm{Dim}\big(V_a(\omega_i)\big)$ for any $a \in \mathbb{C}$. Therefore, $\mathrm{Dim}\big(W(\lambda)\big) = \mathrm{Dim}(L)$. This implies that $\mathrm{Dim}\big(W(\pi)\big) = \mathrm{Dim}(L)$, and therefore $W(\pi) \cong L$. $\square$

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